



Where  $a$  is surjective, projective and  $a_* \mathcal{O}_V = \mathcal{O}_W$ , (\*)  
 (and  $a$  has connected fibers) and  $b$  is finite.

If  $a$  satisfies (\*), it's called an algebraic fiber space.

Goal: If  $X$  is normal and  $K(X, L) > 0$ , then for  $k \gg 0$ ,  
 $X \dashrightarrow Y_k$  is birational to an alg. fiber space  $X_\infty \rightarrow Y_\infty$ .

Prop:  $f: X \rightarrow Y$  projective surj. morphism,  $X, Y$  normal.

Then  $f$  is an algebraic fiber space  $\Leftrightarrow \mathbb{C}(Y) \subseteq \mathbb{C}(X)$   
 is algebraically closed.

Pf:  $\Rightarrow$  If  $\mathbb{C}(Y)$  is not alg. closed in  $\mathbb{C}(X)$ , we can take  
 a finite extension of  $\mathbb{C}(Y)$  in  $\mathbb{C}(X)$  so that  $f$  factors

$$X \xrightarrow{u} Y' \xrightarrow{v} Y$$

where  $v$  is generically finite of  $\deg \geq 2$ . But then fibers of  
 $f$  aren't connected.

$\Leftarrow$  Assume  $\mathbb{C}(Y)$  is algebraically closed in  $\mathbb{C}(X)$ . Let

$$X \xrightarrow{\quad \searrow \quad} Y' \rightarrow Y \quad \text{be the Stein factorization of } f.$$

Then  $Y' \rightarrow Y$  must have  $\deg 1$ , but  $Y$  is normal, so by  
 Zariski's Main Theorem  $Y' \rightarrow Y$  is an isomorphism.  $\square$

Rmk:  $f: X \rightarrow Y$  alg. fiber space,  $L$  a l.b. on  $Y$ .

$$\begin{aligned} \text{Then } H^0(X, f^*L^{\otimes m}) &= H^0(Y, f_*\mathcal{O}_X \otimes L^{\otimes m}) \\ &= H^0(Y, L^{\otimes m}) \end{aligned}$$

So ...

a.)  $K(Y, L) = K(X, f^*L)$ , and

b.)  $f^*: \text{Pic } Y \rightarrow \text{Pic } X$  is injective

(since if  $f^*B = \mathcal{O}_X$ , then  $H^0(f^*B) = H^0(\mathcal{O}_X) = H^0(f^*B^*)$   
 $\neq 0$

$$\Rightarrow B = \mathcal{O}_Y$$

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Lemma:  $L$  semiample,  $X$  normal, projective. Fix  $m \in M(L)$ .

Then for all  $k \gg 0$ ,  $\varphi_m$  has Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{km}} & Y_{km} \xrightarrow{\nu_k} & Y_m \\ & \searrow & \nearrow & \nearrow \\ & & & \varphi_m \end{array} \quad \text{for some finite } \nu_k.$$

i.e.  $\varphi_{km}$  is an alg. fiber space. (and  $Y_{km}, \varphi_{km}$  indep. of  $k$  for  $k \gg 0$ )

Pf:  $X \rightarrow \mathbb{P}^m$  via  $|L^{\otimes m}|$

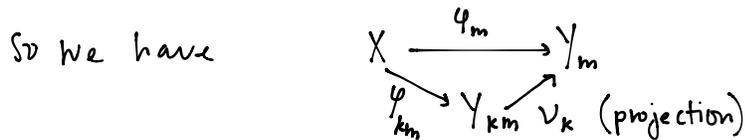
Then  $H^0(\mathcal{O}_{\mathbb{P}^m}(1)) \rightarrow L^{\otimes m}$ , isom. on global sections (same  $\mathbb{P}$ )

$\Rightarrow$  Taking  $\text{Sym}^k H^0(\mathcal{O}_{\mathbb{P}^1}(k)) \rightarrow L^{\otimes km}$  determines a sublinear system

corr. to Veronese re-embedding of  $Y_m$ ,

(possibly not surj. on global sections)

i.e.  $Y_{km} \twoheadrightarrow Y_m$  via projection.



Suppose  $\varphi_m$  contracts a curve  $C$ .

Then  $m(L \cdot C) = 0 = km(L \cdot C)$ , so

$\varphi_{km}$  contracts the curve as well  $\Rightarrow \nu_k$  is finite.

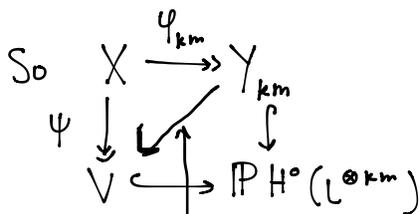
Now assume  $X \xrightarrow{\psi} V \xrightarrow{\mu} Y_m$  is the Stein factorization of  $\varphi_m$

Let  $A_m$  be the ample bundle on  $Y_m$  that pulls back to  $L^{\otimes m}$  on  $X$ .

$\mu$  finite  $\Rightarrow$  It doesn't contract subvarieties  $\Rightarrow B := \mu^* A_m$  is ample.

$\Rightarrow B^{\otimes k}$  v. ample for  $k \gg 0$ .

$$\psi^* B^{\otimes k} = L^{\otimes km} \Rightarrow H^0(X, L^{\otimes km}) = H^0(V, B^{\otimes k})$$



↳ This is surjective  $\Rightarrow$  isom (V and Y are reduced)

$\Rightarrow \Psi = \Psi_{k,m}$ , and since  $\Psi$  is surjective,  $v_k = \mu$ .  $\square$

Semiample fibration theorem: X normal, proj. variety, L semiample.

(i.) There's an alg. fiber space

$$\varphi: X \rightarrow Y$$

s.t. for  $0 < m \in M(L)$ ,  $Y_m = Y$  and  $\varphi_m = \varphi$ .

(ii.)  $\exists$  ample A on Y s.t.  $\varphi^*A = L^{\otimes f}$  ( $f = f(L)$ ).

Pf: WLOG, we can replace L by  $L^{\otimes f}$  so that  $f(L) = 1$ .

Fix  $p, q$  rel. prime and sufficiently large so that (by lemma)

$$Y_{kp} = Y_p \text{ and } \varphi_{kp} = \varphi_p \text{ (resp. } q) \quad \forall k \geq 1.$$

Then  $Y_p = Y_{pq} = Y_q$  and  $\varphi_p = \varphi_{pq} = \varphi_q$ .

Denote  $\varphi: X \rightarrow Y$  the corr. fiber space.

Let  $A_p$  and  $A_q$  be the v. ample bundles on Y s.t.

$$\varphi^*A_p = L^{\otimes p}, \text{ and } \varphi^*A_q = L^{\otimes q}$$

$p, q$  rel. prime  $\Rightarrow 1 = rp + sq$  for some  $r, s \in \mathbb{Z}$ .

Define  $A = A_p^{\otimes r} \otimes A_q^{\otimes s}$ .

Then  $\varphi^* A = L^{\otimes (rp+sq)} = L$

Since  $\varphi$  is a fiber space  $\varphi^* : \text{Pic}(Y) \rightarrow \text{Pic}(X)$  is injective,

so  $A_p = A^{\otimes p}$  and  $A_q = A^{\otimes q}$ , so  $A$  is ample.  $\Rightarrow$  (ii.)

WTS  $Y_m = Y$ ,  $\varphi_m = \varphi \quad \forall m \gg 0$ .

Fix  $c, d \geq 1$ .

Then  $S^c H^0(A^{\otimes p}) \rightarrow A^{\otimes cp}$  and

$S^d H^0(A^{\otimes q}) \rightarrow A^{\otimes dq}$  so

$S^c H^0(A^{\otimes p}) \otimes S^d H^0(A^{\otimes q})$  (re-embedding of  $Y$ ) determines a b.p.f. linear subspace of

$$H^0(A^{\otimes (cp+dq)}) = H^0(L^{\otimes (cp+dq)})$$

so just as before,  $\varphi$  factors

$$\begin{array}{ccc} \text{as } X & \xrightarrow{\varphi} & Y \\ \varphi_{cp+dq} \downarrow & & \uparrow \text{finite (projection)} \\ & & Y_{cp+dq} \end{array}$$

$\varphi$  a fiber space, so it has connected fibers  $\Rightarrow \varphi = \varphi_{cp+dq}$ .

Any  $m \gg 0$  is of the form  $cp+dq$ .  $\square$

Cor:  $X$  normal, projective,  $L$  globally generated. Then  $\exists m_0$  s.t.

$$H^0(L^{\otimes a}) \otimes H^0(L^{\otimes b}) \rightarrow H^0(L^{\otimes a+b}) \text{ is surjective}$$

for  $a, b \geq m_0$ .