Semiample line bundles

X proper scheme, L a line bundle on X

L is <u>semiample</u> if $L^{\otimes m}$ is globally generated for some $m > 0$.
i.e. $\mathbb{B}(L) = \emptyset$

Define $M(L) = \{m \in \mathbb{N} \mid L^{\otimes m} \text{ b.p.f}\} \subseteq \mathbb{N}(L)$.
and $f = f(L) = $ g.c.d. of $M(L)$.

$m \in M(L)$,
$$\varphi_m = \varphi_{|L^{\otimes m}|} : X \longrightarrow \underset{\substack{\uparrow \\ \text{image}}}{Y_m} \subseteq \mathbb{P}H^0(L^{\otimes m})$$

(Consider φ_m to be a map to Y_m instead of $\mathbb{P}$)

<u>Ex</u>: C a curve. L is semi-ample $\Leftrightarrow$ $\deg L > 0$ or L is torsion.

<u>Thm</u> (Zariski): L a line bundle on projective X s.t. $\mathrm{Bs}(|L|)$ is finite. Then L is semi-ample.

<u>Cor</u>: X sm. surface, L l.b. st. $H^0(L) \neq 0$. Then $L(-F)$ is semi-ample for some effective F.

<u>Recall</u>: $g : V \to W$ surj. projective morphism of irr. varieties.

<u>Stein factorization</u>: $V \xrightarrow{a} W' \xrightarrow{b} W$

where a is surjective, projective and $a_* \mathcal{O}_V = \mathcal{O}_{W}$, $\quad (*)$
(and a has connected fibers) and b is finite.

If a satisfies $(*)$, it's called an <u>algebraic fiber space</u>.

<u>Goal</u>: If X is normal and $\kappa(X, L) > 0$, then for $k \gg 0$, $X \dashrightarrow Y_k$ is birational to an alg. fiber space $X_\infty \longrightarrow Y_\infty$.

<u>Prop</u>: $f: X \longrightarrow Y$ projective surj. morphism, X, Y normal. Then f is an algebraic fiber space $\Longleftrightarrow$ $\mathbb{C}(Y) \subseteq \mathbb{C}(X)$ is algebraically closed.

<u>Pf</u>: $\Longrightarrow$ If $\mathbb{C}(Y)$ is not alg. closed in $\mathbb{C}(X)$, we can take a finite extension of $\mathbb{C}(Y)$ in $\mathbb{C}(X)$ so that f factors

$$X \overset{u}{\dashrightarrow} Y' \overset{v}{\dashrightarrow} Y$$

where v is generically finite of deg ≥ 2. But then fibers of f aren't connected.

$\Longleftarrow$ Assume $\mathbb{C}(Y)$ is algebraically closed in $\mathbb{C}(X)$. Let

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "f"'] & Y' \arrow[r] & Y \end{tikzcd}$$

be the Stein factorization of f.

Then $Y' \longrightarrow Y$ must have deg 1, but Y is normal, so by Zariski's Main Theorem $Y' \longrightarrow Y$ is an isomorphism. $\square$

Rmk: $f : X \longrightarrow Y$ alg. fiber space, L a l.b. on Y.

Then $H^0(X, f^*L^{\otimes m}) = H^0(Y, f_*\mathcal{O}_X \otimes L^{\otimes m})$

$$= H^0(Y, L^{\otimes m})$$

So...

a.) $K(Y, L) = K(X, f^*L)$, and

b.) $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ is injective

(since if $f^*B = \mathcal{O}_X$, then $H^0(f^*B) = \underset{\neq 0}{H^0(\mathcal{O}_X)} = H^0(f^*B^*)$

$$\Rightarrow B = \mathcal{O}_Y$$

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Lemma: L semiample, X normal, projective. Fix $m \in M(L)$. Then for all $k >> 0$, φ_m has Stein factorization

$$\begin{tikzcd} X \arrow[r, "\varphi_{km}"] \arrow[rr, bend right, "\varphi_m"'] & Y_{km} \arrow[r, "\nu_k"] & Y_m \end{tikzcd}$$

for some finite ν_k.

i.e. φ_{km} is an alg. fiber space. (and Y_{km}, φ_{km} indep. of k for $k>>0$)

Pf: $X \longrightarrow \mathbb{P}$ via $|L^{\otimes m}|$

Then $H^0(\mathcal{O}_{\mathbb{P}}(1)) \twoheadrightarrow L^{\otimes m}$, isom. on global sections (some $\mathbb{P}$)

$\Rightarrow$ Taking Sym^k; $H^0(\mathcal{O}_{\mathbb{P}}(k)) \twoheadrightarrow L^{\otimes km}$ determines a sublinear system

corr. to Veronese re-embedding of Y_m,

(possibly not surj. on global sections)

i.e. $Y_{km} \twoheadrightarrow Y_m$ via projection.

So we have

$$\begin{tikzcd} X \arrow[rr, "\varphi_m"] \arrow[dr, "\varphi_{km}"'] & & Y_m \\ & Y_{km} \arrow[ur, "\nu_k"'] & \end{tikzcd} \quad \text{(projection)}$$

Suppose φ_m contracts a curve C.

Then $m(L \cdot C) = 0 = km(L \cdot C)$, so

φ_{km} contracts the curve as well $\Rightarrow$ ν_k is finite.

Now assume $X \xrightarrow{\psi} V \xrightarrow{\mu} Y_m$ is the Stein factorization of φ_m

Let A_m be the ample bundle on Y_m that pulls back to $L^{\otimes m}$ on X.

μ finite $\Rightarrow$ It doesn't contract subvarieties $\Rightarrow$ $B := \mu^* A_m$ is ample.

$\Rightarrow B^{\otimes k}$ v. ample for $k \gg 0$.

$\psi^* B^{\otimes k} = L^{\otimes km} \Rightarrow H^0(X, L^{\otimes km}) = H^0(V, B^{\otimes k})$

So
$$\begin{tikzcd} X \arrow[r, two heads, "\varphi_{km}"] \arrow[d, two heads, "\psi"'] & Y_{km} \arrow[d, hook] \arrow[dl] \\ V \arrow[r, hook] & \mathbb{P}H^0(L^{\otimes km}) \arrow[ul] \end{tikzcd}$$

This is surjective $\Rightarrow$ isom (V and Y are reduced)

$\Rightarrow \psi = \varphi_{km}$, and since ψ is surjective, $\nu_k = \mu$. $\square$

<u>Semiample fibration theorem</u>: X normal, proj. variety, L semiample.

(i.) There's an alg. fiber space

$$\varphi : X \longrightarrow Y$$

s.t. for $0 << m \in M(L)$, $Y_m = Y$ and $\varphi_m = \varphi$.

(ii.) $\exists$ ample A on Y s.t. $\varphi^* A = L^{\otimes f}$ ($f = f(L)$).

<u>Pf</u>: WLOG, we can replace L by $L^{\otimes f}$ so that $f(L) = 1$.

Fix p, q rel. prime and sufficiently large so that (by lemma) $Y_{kp} = Y_p$ and $\varphi_{kp} = \varphi_p$ (resp. q) $\forall$ $k \geq 1$.

Then $Y_p = Y_{pq} = Y_q$ and $\varphi_p = \varphi_{pq} = \varphi_q$.

Denote $\varphi : X \longrightarrow Y$ the corr. fiber space.

Let A_p and A_q be the v. ample bundles on Y s.t. $\varphi^* A_p = L^{\otimes p}$, and $\varphi^* A_q = L^{\otimes q}$

p, q rel. prime $\Rightarrow$ $1 = rp + sq$ for some $r, s \in \mathbb{Z}$.

Define $A = A_p^{\otimes r} \otimes A_q^{\otimes s}$.

Then $\varphi^* A = L^{\otimes(rp+sq)} = L$

Since φ is a fiber space $\varphi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective,

so $A_p = A^{\otimes p}$ and $A_q = A^{\otimes q}$, so A is ample. $\Rightarrow$ (ii)

WTS $Y_m = Y$, $\varphi_m = \varphi$ $\forall$ $m \gg 0$.

Fix $c, d \geq 1$.

Then $S^c H^0(A^{\otimes p}) \twoheadrightarrow A^{\otimes cp}$ and

$S^d H^0(A^{\otimes q}) \twoheadrightarrow A^{\otimes dq}$ so

$\underset{\text{(reembedding of } Y)}{S^c H^0(A^{\otimes p})} \otimes S^d H^0(A^{\otimes q})$ determines a b.p.f. linear subseries of

$H^0(A^{\otimes(cp+dq)}) = H^0(L^{\otimes(cp+dq)})$

So just as before, φ factors

as
$$\begin{tikzcd} X \arrow[rr, two heads, "\varphi"] \arrow[dr, "\varphi_{cp+dq}"'] & & Y \\ & Y_{cp+dq} \arrow[ur, "\text{finite (projection)}"'] & \end{tikzcd}$$

φ a fiber space, so it has connected fibers $\Rightarrow \varphi = \varphi_{cp+dq}$.

Any $m \gg 0$ is of the form $cp+dq$. $\square$

<u>Cor</u>: X normal, projective, L globally generated. Then $\exists$ m_0 s.t.

$$H^0(L^{\otimes a}) \otimes H^0(L^{\otimes b}) \longrightarrow H^0(L^{\otimes a+b}) \text{ is surjective}$$

for $a, b \geq m_0$.